**Pedagogical Proofs: Simplifying the Normative Arguments for Majority Rule**

 Robert Dahl argues that May’s theorem, the Condorcet jury theorem, and the Rae-Taylor theorem are three of the four best arguments for majority rule (Dahl 1989, 135-153). The three theorems are the only three arguments Christian List provides for majority rule in an overview of social choice theory (List 2013). Given these theorems’ overwhelming importance in defending majority rule in democratic theory and political philosophy, it seems an important task to ensure, insofar as possible, that the proofs to the theorems are accessible to social choice non-specialists. Furthermore, simplified proofs can make the theorems more easily accessible for graduate education, undergraduate education, policymakers, and the general public. Finally, simpler proofs of fundamental social choice results makes more sophisticated social choice research more interesting for and sought out by non-social choice theorists.

 In the next three sections, I provide simplified proofs of May’s theorem (Section 1), the List-Goodin jury theorem (Section 2), and the Rae-Taylor theorem (Section 3). (Note, the List-Goodin jury theorem is a multiple alternatives generalization of the Condorcet jury theorem on two alternatives.) The proofs make use of visuals (such as illustrations, dice, coins, and urns) to improve pedagogy, though such visuals are not technically necessary for the proofs. At this point, I wish provide rough overviews of the normative importance of the theorems for majority and absolute majority rule.[[1]](#footnote-1)

 May’s theorem roughly shows that if there are two alternatives (say *x* and *y*), and each voter is allowed to express one of three ballots (i.e. *x* over *y*, *y* over *x*, or indifference), then majority rule is uniquely characterized by four normatively desirable conditions: decisiveness, anonymity, neutrality, and positive responsiveness. Roughly speaking the normative reasoning behind the four conditions are as follows. Decisiveness requires the voting system to always have a decision. Anonymity requires each alternative to be treated equally. Neutrality is desirable as it requires the voting system to treat each alternative equally. Positive responsiveness ensures that a voter voting for *x* does not cause *x* to lose, and if there is a tie, an additional vote for *x* breaks such a tie in favor of *x*.

 Suppose there are two alternatives, one of which is true the other of which is false. Roughly, the Condorcet’s jury theorem provides a set of background conditions under which the probability that absolute majority rule correctly chooses the true alternative approaches 1 as the number of voters increases.

 The Rae-Taylor theorem specifies a set of background conditions under which absolute majority rule maximizes the utility of a voter.

**1. May’s Theorem**

1.1 Overview

 First, we provide general definitions, which is followed by definitions of anonymized versions of decisiveness, neutrality, and positive responsiveness. Finally, we define majority rule and state May’s theorem before proceeding to its brief visual proof.

1.2 Definitions

 Let there be two alternatives, *x* and *y*, and let there be *n* voters, where *n* is a positive integer. The *ballot* that voter *i* submits is *bi*. Each voter submits one of three ballots. If *i* submits the ballot “*x* preferred over *y*”, then *bi* = 1. If she submits “indifference between *x* and *y*”, then *bi* = 0. If she submits “*y* preferred over *x*”, then *bi* = -1. A *social welfare function* *f* takes a *profile*, (*b*1, *b*2, …, *bn*) = *p*, and outputs a ballot *b*, which is either 1, 0, or -1.

1.3 Anonymized Decisiveness

 For the profile *p*,let *n*1 be the number of voters that prefer *x* over *y* (where *n*1 ≥ 0), let *n*0 be the number of voters that are indifferent (where *n*0 ≥ 0), and let *n*-1 be the number of voters that prefer *y* over *x* (where *n*-1 ≥ 0), where *n*1 + *n*0 + *n*-1 = *n*. Similarly for the profile *p*′, define *n*1′, *n*0′, and *n*-1′, where *n*1′ + *n*0′ + *n*-1′ = *n*.

*Anonymized Decisiveness*: If *n*1 = *n*1′ and *n*0 = *n*0′ and *n*-1 = *n*-1′, then *f*(*p*) = *f*(*p*′) = {-1, 0, 1}.

 Anonymized decisiveness allows us to reduce profiles to cells. A *cell* specifies the numbers of voters in a profile that prefer *x*, prefer *y*, and are indifferent. For examples, the cell of *p* is (*n*1, *n*0, *n*-1) and the cell of *p*′ is (*n*1′, *n*0′, *n*-1′). If a social welfare function satisfies anonymized decisiveness, then *f*(*p*) = *f*(*n*1, *n*0, *n*-1). Henceforth, unless otherwise stated, all social welfare functions referred to in Section 1 will be assumed to satisfy anonymized decisiveness.

 When a social welfare function satisfies anonymized decisiveness and *n* is fixed, it allows us to visually express the social welfare function with a matrix of anonymized profiles (map). A *map* of an anonymously decisive social welfare function for a given *n* specifies that social welfare function’s output for each logically possible cell given *n*. For example, Figure 1 represents a map for *n* = 5.

**Figure 1: An Example Map for *n* = 5**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|   | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | *f*(0,5,0) | *f*(1,4,0) | *f*(2,3,0) | *f*(3,2,0) | *f*(4,1,0) | *f*(5,0,0) |  |
| **1** | *f*(0,4,1) | *f*(1,3,1) | *f*(2,2,1) | *f*(3,1,1) | *f*(4,0,1) |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | *f*(2,1,2) | *f*(3,0,2) |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

The top row lists all possible values of *n*1 given *n* = 5. The leftmost column lists all possible values of *n*-1 given *n* = 5. Each cell is represented by the intersection of a *n*1 value (i.e. a column) and a *n*-1 value (i.e. a row). The entry in each cell represents the ballot that *f* outputs for that cell.

1.4 Anonymized Neutrality

*Anonymized Neutrality*: If *n*1 = *n*-1′ and *n*0 = *n*0′ and *n*-1 = *n*1′, then *f*(*n*1, *n*0, *n*-1) = -[*f*(*n*1′, *n*0′, *n*-1′)].

 In other words, anonymized neutrality requires that in the map, the entry in the transpose of the cell (*n*1, *n*0, *n*-1) should be the negation of the entry in (*n*1, *n*0, *n*-1). An example is given in Figure 2.

**Figure 2: An Example of Anonymized Neutrality**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|   | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** |  |  |  |  |  | 0 |  |
| **1** |  |  |  | -1 |  |  |  |
| **2** |  |  |  |  |  |  |  |
| **3** |  | 1 |  |  |  |  |  |
| **4** |  |  |  |  |  |  |  |
| **5** | 0 |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

For example, if *f* is anonymously neutral and *f*(1,1,3) = 1 and *f*(5,0,0) = 0, then *f*(3,1,1) = -1 and *f*(0,0,5) = 0.

1.5 Anonymized Positive Responsiveness

*Anonymized Positive Responsiveness*: If *f*(*n*1, *n*0, *n*-1) = {0,1}, then [if (*n*1+1, *n*0 -1, *n*-1) exists on the map, then *f*(*n*1+1, *n*0 -1, *n*-1) = 1] and [if (*n*1+1, *n*0, *n*-1-1) exists on the map, then *f*(*n*1+1, *n*0, *n*-1-1) = 1] and [if (*n*1, *n*0+1, *n*-1-1) exists on the map, then *f*(*n*1, *n*0+1, *n*-1-1) = 1]

In other words, if *f* is anonymously positively responsive and *f*(*n*1, *n*0, *n*-1) is zero or one, then the cell directly above (*n*1, *n*0, *n*-1) and the cell directly to the right of (*n*1, *n*0, *n*-1) and the cell to the upper right of (*n*1, *n*0, *n*-1) must output one. Figure 3 provides an example.

**Figure 3: An Example of Anonymized Positive Responsiveness**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|   | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** |  |  |  |  |  |  |  |
| **1** | 1 | 1 |  |  |  |  |  |
| **2** | {0, 1} | 1 |  |  |  |  |  |
| **3** |  |  |  |  |  |  |  |
| **4** |  |  |  |  |  |  |  |
| **5** |  |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

For example, if *f* is anonymously positively responsive and *f*(0,3,2) is zero or one, then *f*(0,4,1) = *f*(1,3,1) = *f*(1,2,2) = 1.

1.6 Majority Rule and May’s Theorem

*Majority Rule*: If *n*1 > *n*-1, then *f*(*n*1, *n*0, *n*-1) = 1. If *n*1 = *n*-1, then *f*(*n*1, *n*0, *n*-1) = 0. If *n*1 < *n*-1, then *f*(*n*1, *n*0, *n*-1) = -1

*Statement of May’s Theorem*:[[2]](#footnote-2) If there are two alternatives and *n* voters and each voter is allowed to submit any of the three ballots and the social welfare function *f* satisfies the anonymized versions of decisiveness, neutrality, and positive responsiveness, then *f* is majority rule.

**Figure 4: A Visual Proof of May’s Theorem**

Assume *n* = 5 and any voter is allowed to submit any of the three ballots and that *f* is anonymously decisive. Thus the map in Figure 4.1 must be true.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.1** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | *f*(0,5,0) | *f*(1,4,0) | *f*(2,3,0) | *f*(3,2,0) | *f*(4,1,0) | *f*(5,0,0) |  |
| **1** | *f*(0,4,1) | *f*(1,3,1) | *f*(2,2,1) | *f*(3,1,1) | *f*(4,0,1) |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | *f*(2,1,2) | *f*(3,0,2) |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

Applying anonymized neutrality to the map in Figure 4.1, we get the map in Figure 4.2. This is because if *n*1 = *n*-1, then the cell is its own transpose. The entry in a cell cannot be both 1 and -1, but the negation of 0 is 0. Thus if *n*1 = *n*-1, then *f*(*n*1, *n*0, *n*-1) = 0.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.2** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | 0 | *f*(1,4,0) | *f*(2,3,0) | *f*(3,2,0) | *f*(4,1,0) | *f*(5,0,0) |  |
| **1** | *f*(0,4,1) | 0 | *f*(2,2,1) | *f*(3,1,1) | *f*(4,0,1) |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | 0 | *f*(3,0,2) |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

Iterative application of anonymized positive responsiveness leads to Figures 4.3.1, 4.3.2, and 4.3.3.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.3.1** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | 0 | 1 | 1 | *f*(3,2,0) | *f*(4,1,0) | *f*(5,0,0) |  |
| **1** | *f*(0,4,1) | 0 | 1 | 1 | *f*(4,0,1) |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | 0 | 1 |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.3.2** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | 0 | 1 | 1 | 1 | 1 | *f*(5,0,0) |  |
| **1** | *f*(0,4,1) | 0 | 1 | 1 | 1 |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | 0 | 1 |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.3.3** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | 0 | 1 | 1 | 1 | 1 | 1 |  |
| **1** | *f*(0,4,1) | 0 | 1 | 1 | 1 |  |  |
| **2** | *f*(0,3,2) | *f*(1,2,2) | 0 | 1 |  |  |  |
| **3** | *f*(0,2,3) | *f*(1,1,3) | *f*(2,0,3) |  |  |  |  |
| **4** | *f*(0,1,4) | *f*(1,0,4) |  |  |  |  |  |
| **5** | *f*(0,0,5) |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

Application of anonymized neutrality to Figure 4.3.3 leads to the map in Figure 4.4

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **Figure 4.4** | **0** | **1** | **2** | **3** | **4** | **5** | ***n*1** |
| **0** | 0 | 1 | 1 | 1 | 1 | 1 |  |
| **1** | -1 | 0 | 1 | 1 | 1 |  |  |
| **2** | -1 | -1 | 0 | 1 |  |  |  |
| **3** | -1 | -1 | -1 |  |  |  |  |
| **4** | -1 | -1 |  |  |  |  |  |
| **5** | -1 |  |  |  |  |  |  |
| ***n*-1** |  |  |  |  |  |  |  |

But this is just majority rule for *n* = 5. The proof is easily generalizable to any *n*. ■

**2. List-Goodin Jury Theorem**

2.1 Background Conditions of the List-Goodin Jury Theorem

 Let there be *n* voters and *m* alternatives where both are positive integers. Let there be an *m* sided die,[[3]](#footnote-3) where side *d*1 represents alternative *a*1, *d*2 represents *a*2, …, and *dm* represents *am*. Let *πx* be the probability that side *dx* is thrown when the die is rolled. Note, exactly one side is thrown whenever the die is rolled and *π*1 + *π*2 + … + *πm* = 1. Without loss of generality, by definition *π*1 > *π*2 and if *x* < *y* then *πx* ≥ *πy*. Each voter rolls the die, and whichever side she throws corresponds to the alternative she prefers over the remaining *m* – 1 alternatives. Note, each of these rolls is mutually independent. (In other words, the *n* rolls are independent and identically distributed trials.) Let *nx* be the number of voters who throw *dx*.

*Absolute Plurality Rule*: Each voter chooses exactly one of the *m* alternatives which she prefers over the remaining alternatives. If for every alternative *ay*, where *ay* ≠ *ax*, it is the case that *nx* ≥ *ny*, then *ax* is a plurality winner.

*Statement of the List-Goodin Jury Theorem*:[[4]](#footnote-4) Label *a*1 as the true alternative, and label the remaining *m* – 1 alternatives as false. Given the background conditions of the List-Goodin jury theorem, as *n* → ∞, the probability that absolute plurality voting chooses the true alternative as the unique plurality winner converges to 1.

2.2 Proof (List-Goodin Jury Theorem)

The proof ultimately relies on the law of large numbers. Note, the frequentist definition of the probability of *dx* being thrown on a given roll is *πx* = *nx*/*n* as *n* → ∞, where *πx* is the probability of the event (i.e. the probability that *dx* is thrown)and *n* is the number of trials (i.e. *n* voters each roll a die) and *nx* is the number of times the event occurs (i.e. *nx* is the number of times *dx* is thrown by the *n* voters). Given the background conditions of the List-Goodin jury theorem:

Line 1: *π*1 > *π*2 ≥ … ≥ *πm*

Substituting each *πx* with its respective frequentist definition:

Line 2: $\lim\_{n \to \infty } $*n*1/*n* > $\lim\_{n \to \infty } $*n*2/*n* ≥ … ≥ $\lim\_{n \to \infty } $*nm*/*n*

Dividing by *n*:

Line 3: $\lim\_{n \to \infty } $*n*1 > $\lim\_{n \to \infty } $*n*2 ≥ … ≥ $\lim\_{n \to \infty } $*nm*

But if Line 3 is true, then as *n* → ∞, the likelihood that *a*1, the true alternative, is the unique plurality winner of absolute plurality rule, converges on 1. ■

*Condorcet Jury Theorem*: This theorem is the List-Goodin jury theorem for the special case where *m* = 2.

**3. Rae-Taylor Theorem**

3.1 Background Conditions of the Rae-Taylor Theorem

 You are one of *n* voters where *n* is a positive integer. The voters are confronted with a proposition and each voter can submit exactly one of two choices: yes or no. To decide their vote, each voter has a coin, where one side is labeled “yes” and the other labelled “no”. The coin you have is gold colored while the remaining *n* - 1 coins are silver colored.When tossed, each coin has the same probability *π* of landing yes, and the same probability 1 – *π* of landing no, where 0 < *π* < 1. A referee takes the *n* coins, and away from the voters, she tosses each coin once. All the tosses are mutually independent. If a coin lands yes, then she places it in an opaque “yes” urn; if a coin lands no, she places it in an opaque “no” urn. After tossing all *n* coins, she counts the number of coins in the yes urn, *k*, and the number in the no urn, *n* - *k*. The referee approaches you, tells you the values of *k* and *n*-*k*, and asks you to choose exactly one of the two urns. The choice represented by the urn you choose (i.e. yes or no) is implemented. If your golden coin is in the chosen urn, then your utility is one, but if your golden coin is in the unchosen urn, then your utility is zero. To maximize your utility, which urn should you choose?[[5]](#footnote-5)

*Absolute Majority Rule*: A special case of absolute plurality rule for *m* = 2.

*Statement of the Rae-Taylor Theorem*:[[6]](#footnote-6) Given the background conditions of the Rae-Taylor theorem, to maximize your utility: Choose yes if *k* > *n*- *k*. Choose no if *n* - *k* > *k*. Choose either if *k* = *n* - *k*. Given the background conditions of the Rae-Taylor theorem, absolute majority rule thus maximizes your utility.

3.2 Proof (Rae-Taylor Theorem)

The *n* coins tosses are independent and identically distributed random variables. As such, each of the *n* coin tosses has an equal likelihood (i.e. 1/*n*) of being your golden coin’s toss after the tosses have occurred (and you don’t know which of the *n* coin tosses is your golden coin’s toss). Thus, after being informed of the values *k* and *n* - *k*, the likelihood of the golden coin being in the yes urn is *k*(1/*n*) and the likelihood of being in the no urn is (*n* – *k*)(1/*n*). Thus the utility of choosing the yes urn is (1)( *k*/*n*) + (0)([*n* – *k*]/*n*) = *k*/*n*. Similarly, the utility of choosing the no urn is (*n* – *k*)/*n*. Clearly, to maximize your utility, you should choose yes if *k* > *n*– *k*, choose no if *n* - *k* > *k*, and choose either if *k* = *n* - *k*. Absolute majority rule does that. ■

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1. In the literature, there is often a conflation between majority rule and absolute majority rule. Roughly speaking, majority rule allows abstention while the absolute version does not. [↑](#footnote-ref-1)
2. Technically speaking, May’s theorem is an if-and-only-if claim, not an if-then claim. However, demonstration of the converse claim is trivial (May 1952). [↑](#footnote-ref-2)
3. Technically, the die is unnecessary for our proof as it is only used to help illustrate the underlying random processes, but it simplifies pedagogical exposition. [↑](#footnote-ref-3)
4. Technically, this is a proof of the List-Goodin proposition 2 (List and Goodin 2001, 285). [↑](#footnote-ref-4)
5. Keep in mind the coins-and-urns analogy is an illustration of the underlying math. Each voter has a probability *π* of voting yes (e.g. probability *π* of landing yes). How each voter votes is mutually independent of how other voters vote (e.g. mutually independent of other coin tosses). You are told how many voters voted yes and how many voted no, but you are not told who voted for whom (e.g. how many coins are in each urn, but you are not informed who tossed which coin). Which choice maximizes the likelihood you are in the winning coalition (e.g. which urn should you choose to maximize your likelihood of your coin being in the chosen urn)? [↑](#footnote-ref-5)
6. Technically speaking, the Rae-Taylor theorem identifies absolute majority rule as a rule that minimizes the expected number of losing coalitions you belong to over a series of independent and identically distributed (i.e. IID) yes-no propositions (Rae 1969; Taylor 1969). But two things. First, because being in a winning coalition is identical to not being in a losing coalition, maximizing your number of winning coalition memberships can be treated the same as minimizing your number of losing coalition memberships. Second, because the propositions are IID, you should have the same choosing strategy for each proposition; thus we can focus on maximizing your likelihood of being in the winning coalition for one proposition and can apply that same strategy to the other propositions. This greatly simplifies the pedagogical exposition of the proof. [↑](#footnote-ref-6)